# Chapter 5 - Choice

→ Now that we've defined what is affordable (budget constraint) and what is preferred, we put these together to see how consumers choose the most preferred bundle from their budget sets

## Optimal choice

→ To solve the consumer's problem, we'll use calculus

→ Note that this problem is constrained

→ Note that

→ consumption expenditures are constrained by the budget set

→ consumption must be non-negative

→ So the general problem can be stated as:

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$\text{subject to } p_1x_1 + p_2x_2 \leq m, \; x_1 \geq 0, \; x_2 \geq 0$$

$(x_1^*, x_2^*)$ will be the consumption bundle that solves this problem

→ We'd seen how to solve an unconstrained maximization problem before

→ ~~set the~~ s

→ find where the slope = 0

→ To solve a constrained problem, we will introduce a new tool - the Lagrangian

→ The Lagrangian

→ a way to incorporate the objective function (e.g $u(x_1, x_2)$) and the constraint(s) into a single function to be maximized

→ we will incorporate the constraints w/ Lagrange multipliers

→ these are constants $\geq 0$ that penalize the Lagrangian function if the constraint is violated

→ this ensures that the constraints are not violated - you can't maximize the Lagrangian w/a violation

Example: Cobb-Douglas utility

$$\max \; x_1^c x_2^d$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq m$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

The Lagrangian is:

$$\mathcal{L} = \underbrace{x_1^c x_2^d}_{\substack{\text{objective}\\\text{function}}} + \underset{\substack{\text{Lagrange}\\\text{multiplier on}\\\text{BC}}}{\lambda}(m - p_1 x_1 - p_2 x_2) + \underbrace{\mu_1 x_1 + \mu_2 x_2}_{\substack{\text{Lagrange multipliers}\\\text{on the}\\\text{non-negativity}\\\text{constraints on}\\\text{consumption}}}$$

→ note how constraints are entered

→ they are entered such that if they are violated, they lower the value of $\mathcal{L}$

→ e.g if spend more than income

$m - p_1 x_1 - p_2 x_2 < 0$

These Lagrange multipliers are all non-negative:

$$\lambda \geq 0$$

$$\mu_1 \geq 0$$

$$\mu_2 \geq 0$$

→ They also have an interpretation that we'll see more clearly in a moment.

→ these multipliers will represent the value of relaxing the constraint

Lagrange's theorem says that the solution to the constrained maximization problem satisfies the following conditions:

These are the Kuhn-Tucker Conditions → necessary conditions for a sol'n to the opt. problem

$$\frac{\partial \mathcal{L}}{\partial x_1} = c x_1^{c-1} x_2^{d} - \lambda p_1 + \mu_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = d x_1^{c} x_2^{d-1} - \lambda p_2 + \mu_2 = 0$$

These are called the "complementary slackness" conditions

$$\lambda \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - m = 0 \text{ or } \lambda = 0$$

$$\mu_1 \frac{\partial \mathcal{L}}{\partial \mu_1} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \mu_1} = x_1 = 0 \text{ or } \mu_1 = 0$$

$$\mu_2 \frac{\partial \mathcal{L}}{\partial \mu_2} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \mu_2} = x_2 = 0 \text{ or } \mu_2 = 0$$

Notice what the necessary conditions are telling us:

→ either don't spend all money (and $\lambda = 0$) or do and $\lambda \geq 0$

→ either consume positive $x_1$ (and $\mu_1 = 0$) or don't (and $\mu_1 > 0$)

→ either consume positive $x_2$ (and $\mu_2 = 0$) or don't (and $\mu_2 = 0$)

⇒ and regarding the first 2 necessary conditions:

→ the slope of the utility function will not equal zero at the maximum if the constraints bind

Solving the constrained optimization problem

→ we have 5 equations and 5 unknowns $(x_1, x_2, \lambda, \mu_1, \mu_2)$

① ⇒ $c x_1^{c-1} x_2^{d} + \mu_1 = \lambda p_1$

② ⇒ $d x_1^{c} x_2^{d-1} + \mu_2 = \lambda p_2$

note that if $x_1 = 0$, then ① ⇒ $\underbrace{\mu_1}_{>0} = \lambda p_1$

⇒ $\lambda > 0$

but then ② ⇒ $\mu_2 > 0$ ⇒ $x_2 = 0$

but then BC not binding

⇒ $\lambda = 0$ →←

→ so $x_1 > 0$, $x_2 > 0$

B/c $x_1 > 0$, $x_2 > 0$, $\mu_1 = \mu_2 = 0$

so ① becomes: $c x_1^{c-1} x_2^{d} = \lambda p_1$

② becomes: $d x_1^{c} x_2^{d-1} = \lambda p_2$

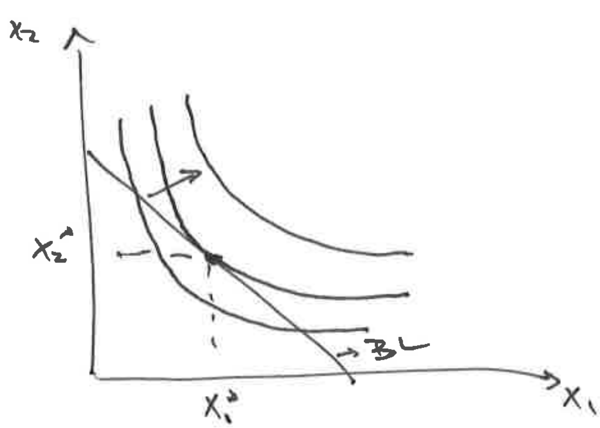
→ dividing ① by ② we get:

$$\frac{c x_1^{c-1} x_2^{d}}{d x_1^{c} x_2^{d-1}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2}$$

$$\Rightarrow \underbrace{\frac{c x_2}{d x_1}}_{\text{MRS}} = \underbrace{\frac{p_1}{p_2}}_{\text{Price Ratio}}$$

What this means → slope of indiff. curve equals slope of budget line!

We've seen this:




→ the highest indifference curve is the one that just touches the BL

→ "just touches" means is tangent to the BL

→ tangent, means has the same slope at that point

→ So the calculus of our FOCs gives the same sol'n we got by looking at the graph

→ we can continue w/ our equations to solve for our demand for $x_1$ and $x_2$:

$$\frac{cx_2}{dx_1} = \frac{p_1}{p_2}$$

$$\Rightarrow \quad \underbrace{x_2 = \frac{p_1}{p_2}\frac{d}{c}x_1}_{x_2(x_1)} \rightarrow \text{put this into BC:}$$

$$p_1x_1 + p_2x_2 = m$$

$$p_1x_1 + \cancel{p_2}\frac{p_1}{\cancel{p_2}}\frac{d}{c}x_1 = m$$

$$p_1x_1\left(1 + \frac{d}{c}\right) = m$$

$$p_1x_1\left(\frac{c+d}{c}\right) = m$$

$$\Rightarrow x_1 = \frac{m}{p_1}\frac{c}{c+d}$$

$$\Rightarrow x_2 = \frac{p_1}{p_2}\frac{d}{c}x_1 = \frac{\cancel{p_1}}{p_2}\frac{m}{\cancel{p_1}}\frac{\cancel{c}}{c+d}\frac{d}{\cancel{c}}$$

$$\Rightarrow x_2 = \frac{m}{p_2}\frac{d}{c+d}$$

→ note demands from Cobb-Douglas utility:

$$x_1 = \frac{m}{p_1} \frac{c}{c+d}$$

$$x_2 = \frac{m}{p_2} \frac{d}{c+d}$$

→ if make monotonic transform of utility function such that

$c+d = 1$:

e.g. let $a = \frac{c}{c+d}$

$$1-a = \left(1 - \frac{c}{c+d}\right) = \frac{c+d}{c+d} - \frac{c}{c+d} = \frac{d}{c+d}$$

so $v(x_1, x_2) = u(x_1, x_2) = \left(x_1^c x_2^d\right)^{\frac{1}{c+d}}$

$$= x_1^{\frac{c}{c+d}} x_2^{\frac{d}{c+d}}$$

$$= x_1^{a} x_2^{1-a}$$

→ this will rep the same preferences

demands will be:

$$x_1 = \frac{m}{p_1} a$$

$$x_2 = \frac{m}{p_2}(1-a)$$

what these mean:

$$\underbrace{p_1 x_1}_{\text{amt spend on } x_1} = \underbrace{a m}_{\text{fraction } a \text{ of income}}$$

$$p_2 x_2 = (1-a) m$$

→ What about the constraints?

→ we solve for $\mu_1 = \mu_2 = 0$

⇒ non-neg constraint doesn't bind

→ we can solve for $\lambda$ from eq'n ① or ②:

e.g. ① ⇒ $c x_1^{c-1} x_2^{d} = \lambda p_1$

$$\underbrace{\frac{c x_1^{c-1} x_2^{d}}{p_1}}_{\text{MU per dollar on } x_1} = \lambda$$

→ not that $\lambda$ also equals the MU per dollar on $x_2$

→ which makes sense ~~if~~ if these amounts weren't the same, the consumer wouldn't be at an optimum → she could spend a little less on one good and more on the other.

Example: Perfect substitutes

$u(x_1, x_2) = x_1 + x_2$

$\Rightarrow \mathcal{L} = x_1 + x_2 + \lambda(m - p_1x_1 - p_2x_2) + \mu_1 x_1 + \mu_2 x_2$

FOCs:

① $\frac{\partial \mathcal{L}}{\partial x_1}: \quad 1 - \lambda p_1 + \mu_1 = 0$

② $\frac{\partial \mathcal{L}}{\partial x_2}: \quad 1 - \lambda p_2 + \mu_2 = 0$

③ $\frac{\partial \mathcal{L}}{\partial \lambda} = p_1x_1 + p_2x_2 - m = 0$ or $\lambda = 0$

④ $\frac{\partial \mathcal{L}}{\partial \mu_1} = x_1 = 0$ or $\mu_1 = 0$

⑤ $\frac{\partial \mathcal{L}}{\partial \mu_2} = x_2 = 0$ or $\mu_2 = 0$

① $\Rightarrow \underbrace{1 + \mu_1}_{>0} = \lambda p_1$

$\Rightarrow \lambda > 0 \Rightarrow$ BC binds (spend all money)

$\Rightarrow x_1 > 0$ or $x_2 > 0$ or both

② $\Rightarrow 1 + \mu_2 = \lambda p_2$

→ Nothing eliminates case that one of demands $= 0$

→ this is called a corner sol'n

What we'll do then is to ~~const~~ consider an interior sol'n then consider the corner sol'ns (all $x_1$ or all $x_2$) and see what conditions on prices put us there:

if $x_1$ and $x_2 > 0$:

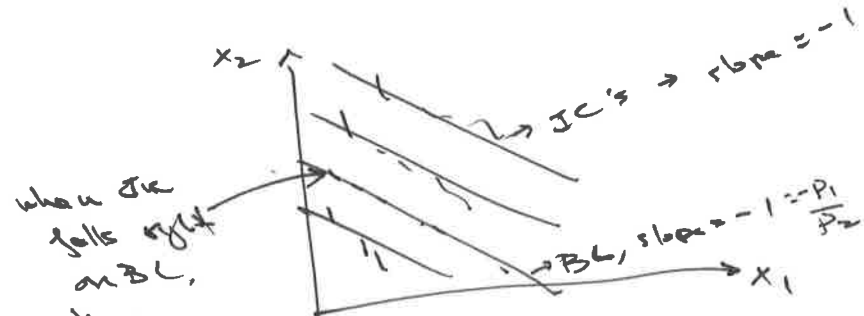
$\Rightarrow \mu_1 = \mu_2 = 0$

$\Rightarrow \quad 1 = \lambda p_1 = \lambda p_2$

$\Rightarrow p_1 = p_2$

$\rightarrow$ so interior sol'n only if $p_1 = p_2$

$$\Rightarrow \underbrace{1}_{MRS} = \underbrace{\frac{p_1}{p_2}}_{PR}$$




$\rightarrow$ consumer indifferent b/w any point on BL

if $x_1 > 0$, $x_2 = 0$

$\Rightarrow \mu_1 = 0, \ \mu_2 > 0$

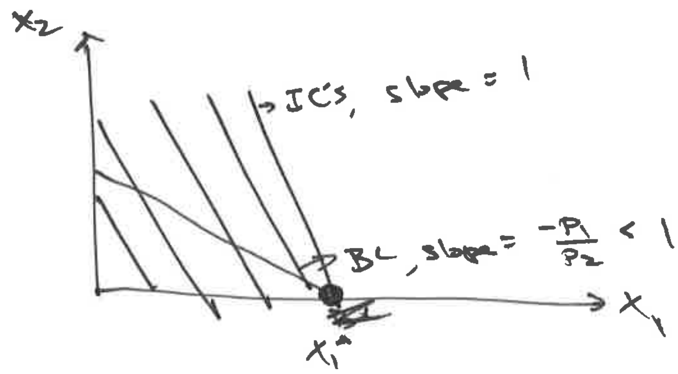
$\Rightarrow \quad 1 = \lambda p_1$

$\underbrace{1 + \mu_2}_{>1} = \lambda p_2$

$\Rightarrow \lambda p_1 < \lambda p_2$

$\Rightarrow p_1 < p_2$

$\rightarrow$ if $p_1 < p_2$, consume only $x_1$:




if $x_2 > 0$, $x_1 = 0$

$\Rightarrow \mu_1 > 0$, $\mu_2 = 0$

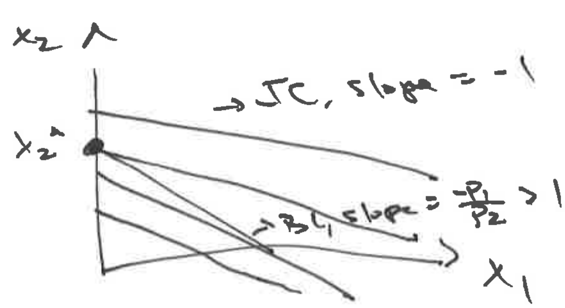
$\Rightarrow \quad 1 + \mu_1 = \lambda p_1$

$1 = \lambda p_2$

$\Rightarrow \quad \cancel{\lambda} p_1 > \cancel{\lambda} p_2$

$p_1 > p_2$

$\rightarrow$ if $p_1 > p_2$, consume only $x_2$



→ consider other cases:

→ Bads

→ Neutrals

→ concave prefs

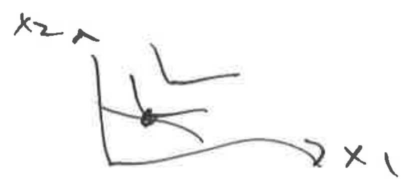
→ The Lagrangian w/ all constraints will always work

→ but can be harder to remember where corner sol'n likely

→ also, case like perfect complements doesn't have derivative:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

→ so think @ graphically...



Utility functions and optimal tax

policies

→ consider 2 taxes

① a quantity tax on good 1 at a rate of $\tau$

② a lump sum tax, $T$

→ lets make the sizes of the taxes the same - so that the revenue raised from the quantity tax at the consumer's optimal choice, $x_1^*$, ~~gives~~ is the same as that raised from the lump sum tax, $T$

$$\Rightarrow \quad T = \tau x_1^*$$

w/ ~~cons~~ quantity tax:

$$\max \; u(x_1, x_2)$$
$$\text{s.t. } (p_1+\tau)x_1 + p_2 x_2 \le m$$

→ assume interior sol'n

$$\Rightarrow \mathcal{L} = u(x_1, x_2) + \lambda\,(m - (p_1+\tau)x_1 - p_2 x_2)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda(p_1+\tau) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda(p_2) = 0$$

$$\Rightarrow \underbrace{\frac{\partial u(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial u(x_1, x_2)}{\partial x_2}}_{\text{MRS}} = \underbrace{\frac{p_1+\tau}{p_2}}_{\text{slope of BC}}$$

w/ lump sum tax:

$$\max \; u(x_1, x_2)$$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \le m - T$$

→ assume interior sol'n

$$\Rightarrow \mathcal{L} = u(x_1, x_2) + \lambda(m - T - p_1 x_1 - p_2 x_2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0$$

$$\Rightarrow \underbrace{\frac{\partial u(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial u(x_1, x_2)}{\partial x_2}}_{MRS} = \underbrace{\frac{p_1}{p_2}}_{\text{slope of BC}}$$

→ since $x_1^*(T)$ affordable under lump sum tax → then lump sum tax make consumer at least as well off